PMM U.S.S.R.,Vol.49,No.6,pp. 688-694,1985
Printed in Great Britain

0021-8928/85 $10.00+0.00
Pergamon Journals Ltd.

ON ESTIMATING THE ATTRACTION REGIONS OF THE EQUILIBRIUM STATES OF DYNAMIC SYSTEMS BY THE DIRECT LYAPUNOV METHOD*

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The method of non-local reduction /1, 2/ is proposed for estimating the attraction regions of asymptotically stable equilibriums states of dynamic systems linked to the construction of Lyapunov functions in which the thoroughly investigated trajectories of two-dimensional systems are contained. The class of functions described below widens the class of functions obtained on the basis of Lur'e-Postnikov functions /3, 4/. That region entirely belongs to the attraction region of the system equilibrium state and is wider than the estimate of the attraction region estimated by the Lur'e-Postnikov function basis. The attraction region may exceed the limits of the interval in which the non-linearity satisfies the generalized Routh-Hurwitz conditions /5/. The algorithm proposed below contains the logarithms for specific conditions based on Lyapunov functions of the Lur'e type and the special condition of V.M. Popov.

The systems of non-linear differential equations considered have the form

$$x^{\cdot} = Px + q\varphi(\sigma), \quad \sigma = r^*x \tag{1}$$

where P is a constant matrix $(n+1)\times(n+1)$ and q, r are $(n+1)$-dimensional constant vectors. The non-linear function $\varphi(\sigma)$ is continuous and unique for all $\sigma \neq 0$. For $\sigma = 0$ the value of $\varphi(\sigma)$ is the set $[\varphi(-0), \varphi(+0)]$. The function $\varphi(\sigma)$ is assumed to satisfy the inequality.

$$\varphi(\sigma)\sigma > 0, \quad \forall \sigma \neq 0 \tag{2}$$

The matrix P has an arbitrary spectrum. When the eigenvalues of P have positive real parts, system (1) will be called inherently unstable.

We formulate the problem as follows. It is required to construct in the phase space of system (1) a region entirely belonging to the attraction region of the equilibrium space $x = 0$, i.e. evaluating from "inside" the region of asymptotic stability "in the large".

The problem of such an estimation of the attraction region of the zero solution of system (1) then reduces to investigating the second-order system

$$\eta^{\cdot} = \alpha\eta - \varphi(\sigma), \quad \sigma^{\cdot} = \eta - \beta\varphi(\sigma); \ \alpha \geqslant 0, \ \beta \geqslant 0 \tag{3}$$

where the function $\varphi(\sigma)$ satisfies condition (2). Along with system (3) we shall consider the equation

$$\frac{dF(\sigma)}{d\sigma} = \frac{\alpha F(\sigma) - \varphi(\sigma)}{F(\sigma) - \beta\varphi(\sigma)} \tag{4}$$

The solution $F(\sigma)$ of Eq.(4) correspond to trajectories of system (3) when $\sigma \neq 0$. The field of directions of system (3) shows that for numbers $\sigma_1 < 0 < \sigma_2$ and $s_0 \geqslant 0$ Eq.(4) has solutions $F_-(\sigma)$, $F_+(\sigma)$ with the following properties:

$$\begin{gathered} F_-'(\sigma) > 0, \quad \forall\sigma \in [\sigma_1, 0), \ F_-(\sigma_1) = s_0 \\ F_+'(\sigma) > 0, \quad \forall\sigma \in (0, \sigma_2], \ F_+(\sigma_2) = -s_0 \end{gathered} \tag{5}$$

To prove Theorem 1 we shall use the solution $F(\sigma)$ of Eq.(4) specified for $\sigma \in [\sigma_1, \sigma_2]$, which is defined as follows:

$$F(\sigma) = \begin{cases} F_-(\sigma), & \sigma \in [\sigma_1, 0] \\ F_+(\sigma), & \sigma \in (0, \sigma_2] \end{cases}$$

We will assume that for some μ the relation

$$0 \leqslant \varphi(\sigma)\sigma \leqslant \mu\sigma^2, \ \forall\sigma \in [\sigma_1, \sigma_2] \tag{6}$$

is satisified.

Consider the behaviour of the trajectories of system (1) close to the hyperplane $\sigma = 0$, assuming that $\rho = r^*q < 0$. In that case system (1) is in the so-called sliding mode in the hyperplane "band" $r^*x = 0$

*Prikl.Matem.Mekhan.,49,6,900-908,1985

$$\Pi = \{x:\ -\rho\varphi(-0) \leqslant r^*Px \leqslant -\rho\varphi(+0),\ r^*x = 0\} \tag{7}$$

in which $\sigma^{\cdot}(t) \equiv 0$. The solution of system (1) can only leave the band at the boundary points of set (7), and the motion in the sliding mode is defined by the linear system /5/

$$x^{\cdot} = (I - qr^*/\rho)\,Px,\ \ r^*x = 0 \tag{8}$$

Consider sets of form $\Psi(\zeta) = \{x: x^*Lx \leqslant \zeta,\ r^*x = 0\}$ of some positive definite matrix $L = L^*$. If the number ζ is such that the set $\Psi(\zeta)$ belongs to the band (7) and the matrix L satisfies the inequality $L(I - \rho^{-1}qr^*)\,P + P^*(I - \rho^{-1}rq^*)\,L \leqslant 0$, the set $\Psi(\zeta)$ is positively invariant for solutions of system (8) and, consequently, for solutions of $x(t)$ of system (1). This means that when $x(0) \in \Psi(\zeta)$, then $x(t) \in \Psi(\zeta)$ for all $t \geqslant 0$. If moreover all roots of the characteristic equation of system (8) /5/

$$\det(P - pI)\,r^*(P - pI)^{-1}q = 0 \tag{9}$$

are to the left of the imaginary axis, the solutions of system (8) approach the point $x = 0$ as $t \to +\infty$.

We will introduce the notation $L_0 = \sup\{\zeta: \Psi(\zeta) \in \Pi\}$ and $\chi(p) = r^*(P - pI)^{-1}q$, and determine for the arbitrary matrix $H = H^* > 0$ and numbers σ_1, σ_2 the number H_0 using the formula

$$H_0 = \min_{i=1,2}\left(\min_{r^*x=\sigma_i} x^*Hx\right)$$

A relative minimum of the quadratic form x^*Hx is reached at $x_1 = \sigma_1 H^{-1}r(r^*H^{-1}r)^{-1}$ /6–9/ on condition that $r^*x = \sigma_1$.

Hence

$$H_0 = \min\left(\frac{\sigma_1^2}{r^*H^{-1}r},\ \frac{\sigma_2^2}{r^*H^{-1}r}\right)$$

Let us formulate the theorem on the estimate of attraction region of the solution $x = 0$ of system (1).

Theorem 1. If real symmetric matrices $H > 0, L > 0$ and numbers $\lambda \leqslant 0, \tau \geqslant 0, \theta > 0, \varepsilon_\lambda < 0$ exist for $\lambda = 0$, $\varepsilon_\lambda = \theta/\lambda$ and $\lambda \neq 0$ such that the following conditions are satisfied:

1) the set $\Omega_0 = \{x: x^*Lx \leqslant \varepsilon_\lambda\lambda\theta^{-1}L_0,\ \ r^*x = 0\}$ is positively invariant for solutions of system (8);

2) $P + \lambda I$ is a Hurwitz matrix;

3) the following matrix inequalities $R \leqslant 0$ and $-\gamma R \geqslant gg^*$, where $R = H(P + \lambda I) + (P^* + \lambda I)\,H + L$, $g = Hq + {}^1/_2(\theta P^*r + \tau r)$, are satisfied

$$\gamma = \tau\mu^{-1} - \rho\theta + \rho\varepsilon_\lambda\lambda$$

4) for the solution $F(\sigma)$ of (4) with $\alpha = -\lambda(\theta\Gamma^{-1})^{1/2}$, $\beta = -\rho\varepsilon_\lambda\lambda(\theta\Gamma)^{-1/2}$, $\Gamma = -(\theta P^*r + \tau r)^*q$, $s_0 = (2\theta^{-1}H_0)^{1/2}$, which has the properties (5) the relation $F(-0) = -F(+0)$ holds, and moreover when $\lambda \neq 0$ the condition

$$F(-0) \leqslant \min\left\{\sqrt{\frac{L_0}{-\lambda\theta}},\ \sqrt{\frac{\beta}{\alpha}}\inf_{\sigma\neq 0}|\varphi(\sigma)|\right\} \tag{10}$$

is satisfied,

5) when in condition (6), $\lambda \neq 0$, we have $\mu = +\infty$.

Then any solution $x(t)$ of system (1) with initial conditions

$$x(0) \in \Omega = \{x: x^*Hx < {}^1/_2\theta F^2(\sigma),\ \sigma \in (\sigma_1, \sigma_2)\} \cup \Omega_0 \tag{11}$$

belongs to the region Ω for all $t \geqslant 0$.

If moreover all the roots of Eq.(9) are on the left of the imaginary axis, the solution $x(t)$ with initial conditions (11) approaches the point $x = 0$ as $t \to +\infty$, i.e. the region Ω belongs to the attraction region of the equilibrium state $x = 0$ of system (1).

The condition for the matrix H to exist defines the following lemma.

Lemma 1. For the existence of the matrix $H = H^*$ that satisfies condition 3) of Theorem 1 it is necessary and sufficient that the inequality

$$\mathrm{Re}\,[(\tau + \theta p)\,\chi(p) - q^*(P^* - pI)^{-1}L\,(P - pI)^{-1}q] + \rho\varepsilon_\lambda\lambda + \tau/\mu \geqslant 0 \tag{12}$$

where $p = i\omega - \lambda$ is satisfied for all $\omega \in R^1$. An algorithm for determining the matrix H is described in /10, 11/).

Proof of Theorem 1. By virtue of condition 3) of Theorem 1, by Lemma 1.2.1 in /5/ the inequality

$$2x^*H\,[(P + \lambda I)\,x + q\xi] + \theta\xi r^*(Px + q\xi) + \tau\xi r^*x + x^*Lx - (\rho\varepsilon_\lambda\lambda + \tau\mu^{-1})\xi^2 \leqslant 0 \tag{13}$$

is satisfied for any x, ξ.

Consider the function

$$V(x) = x^*Hx - {}^1/_2\theta F^2(\sigma) \tag{14}$$

Let $x(t)$ be some solution of system (1) with initial conditions $x(0) \in \Omega \setminus \Omega_0$ (if $x(0) \in \Omega_0$), then $x(t) \in \Omega_0$ for all $t \geqslant 0$ by condition 1) of Theorem 1). Calculating the time derivative of the function $v(t) = V(x(t))$, by virtue of system (1) we obtain

$$v^{\cdot} + 2\lambda v = 2x^*H\,[(P + \lambda I)\,x + q\varphi(\sigma)] - \theta\lambda F^2(\sigma) - \theta F(\sigma)F'(\sigma)\,r^*\,[Px + q\varphi(\sigma)]$$

where $x = x(t)$, $\sigma = \sigma(t)$. Using (13) we obtain

$$v^{\cdot} + 2\lambda v \leqslant -\theta\varphi(\sigma)r^*\,[Px + q\varphi(\sigma)] - \tau\varphi(\sigma)\,r^*x + \tau\mu^{-1}\varphi^2(\sigma) - x^*Lx + \rho\varepsilon_\lambda\lambda\varphi^2(\sigma) - \theta\lambda F^2(\sigma) - \theta F(\sigma)\,F'(\sigma)\,r^*\,[Px + q\varphi(\sigma)] \quad (15)$$

Since $v(0) = x^*(0)\,Hx(0) - {}^1/_2\theta F^2 + (\sigma(0)) < 0$, then from the continuity of the function $v(t)$ it follows that a number T exists such that for $t < T$ the condition $v(t) < 0$ is satisfied, i.e. $x(t) \in \Omega$. Note that

$$\sigma(t) \in (\sigma_1, \sigma_2), \quad \forall t : v(t) < 0 \quad (16)$$

Indeed, since $\sigma(0) \in (\sigma_1, \sigma_2)$, there is a number t_1 such that when $t \in [0, t_1)$ the inclusion $\sigma(t) \in (\sigma_1, \sigma_2)$ holds. Let us assume that $\sigma(t_1) = \sigma_i\ (i = 1, 2)$. Then the relations $v(t_1) < 0$, $F(\sigma_1) = -F(\sigma_2) = \sqrt{2H_0/\theta}$ imply that $x^*(t_1)Hx(t_1) < H_0$ which is at variance with the positive definiteness of the matrix H and the definition of the number H_0.

Assume now that $v(T) = 0$. Let us first consider the case of $\lambda \neq 0$. Then $\varepsilon_\lambda\lambda = \theta$, $\mu = +\infty$, and from the positive definiteness of the matrix L and (15) we have

$$v^{\cdot} + 2\lambda v \leqslant -\theta\,[F(\sigma)\,F'(\sigma) + \varphi(\sigma)]\,r^*Px - \theta\lambda F^2(\sigma) - \rho\theta F(\sigma)\,F'(\sigma)\,\varphi(\sigma) - \tau r^*x\varphi(\sigma) - \tau F(\sigma)\,F'(\sigma)\,r^*x \quad (17)$$

The last term has been added to the right side, since $F(\sigma)\,F'(\sigma)\,\sigma < 0$ holds for all $\sigma \neq 0$. From condition 3) of Theorem 1 it follows that ($\gamma = 0$ when $\lambda \neq 0$), hence

$$Hq = -{}^1/_2\,(\theta P^*r + \tau r) \quad (18)$$

Let us put $c = \theta P^*r + \tau r$, $\Gamma = -c^*q$. The positive definiteness of the matrix H and (18) imply that $\Gamma > 0$. But by Lemma 1.2.5 in /5/ and on the basis of (18) we have the equations

$$\det\,[H + (2c^*q)^{-1}cc^*] = \det H\,[1 + (2c^*q)^{-1}c^*H^{-1}c] = 0$$

from which, by virtue of the positive definiteness of the matrix H and the positiveness of the number Γ. we obtain the inequality $H + (2c^*q)^{-1}cc^* \geqslant 0$ or $x^*Hx \geqslant (2\Gamma)^{-1}\,|\,c^*x\,|^2$. From the last inequality and the condition $v(t) \leqslant 0$, when $t \in [0, T]$ it follows that

$$|\,c^*x\,|^2 \leqslant \Gamma\theta F^2(\sigma) \quad (19)$$

Condition 4) of Theorem 1 ensures the relation

$$F(\sigma)\,[F(\sigma)\,F'(\sigma) + \varphi(\sigma)] \leqslant 0, \quad \forall\sigma \neq 0 \quad (20)$$

This follows from (10), the inequality $|F(\sigma)| \leqslant F(-0)$ when $\sigma \in [\sigma_1, \sigma_2]$, and the equation

$$F(\sigma)\,F'(\sigma) + \varphi(\sigma) = [\alpha F^2(\sigma) - \beta\varphi^2(\sigma)][F(\sigma) - \beta\varphi(\sigma)]^{-1}$$

For $t \leqslant T$ such that $\sigma(t) \neq 0$ we continue the inequality (17), using (19) and (20), and the fact that $\varepsilon_\lambda\lambda = \theta$ when $\lambda \neq 0$. We obtain

$$v^{\cdot} + 2\lambda v \leqslant -[F(\sigma)\,F'(\sigma) + \varphi(\sigma)](\theta r^*P + \tau r^*)\,x - \theta\lambda F^2(\sigma) - \rho\theta F(\sigma)\,F'(\sigma)\,\varphi(\sigma) \leqslant -[F(\sigma)\,F'(\sigma) + \varphi(\sigma)]\,\sqrt{\Gamma\theta}F(\sigma) - \theta\lambda F^2(\sigma) - \rho\varepsilon_\lambda\lambda F(\sigma)F'(\sigma)\,\varphi(\sigma)$$

Selecting the numbers $\alpha = -\lambda\sqrt{\theta\Gamma^{-1}}$, $\beta = -\rho\varepsilon_\lambda\lambda\,(\theta\Gamma)^{-1/2}$, by virtue of (4) we obtain

$$v^{\cdot} + 2\lambda v \leqslant -\sqrt{\Gamma\theta}\,F(\sigma)\,[F(\sigma)\,F'(\sigma) - \alpha F(\sigma) - \beta F'(\sigma) \times \varphi(\sigma) + \varphi(\sigma)] = 0$$

Thus when $\lambda \neq 0$

$$v^{\cdot} + 2\lambda v \leqslant 0, \quad t \in [0, T] : \sigma(t) \neq 0 \quad (21)$$

For $\sigma(t) = 0$, $\sigma^{\cdot}(t) = 0$ (i.e. $r^*x = 0$, $r^*(Px + q\varphi(\sigma)) = 0$) from (15) we have

$$v^{\cdot} + 2\lambda v \leqslant -\lambda\theta F^2(\sigma) - x^*Lx \quad (22)$$

From (22) and (10) it follows that

$$v^{\cdot} + 2\lambda v \leqslant 0 \quad (23)$$

for such t that $\sigma(t) = 0$ and $x^*(t)\,Lx(t) > L_0 \geqslant -\lambda\theta F^2(0)$.

If, however, for some $t = t_2$ we have $\sigma(t_2) = 0$ and $x^*(t_2)\,Lx(t_2) \leqslant L_0$, i.e. $x(t_2) \in \Omega_0$, then $x(t) \in \Omega_0$ for all $t \geqslant t_2$ by virtue of the positive invariance of the set Ω_0. It follows from this and (21) that condition (23) is satisfied for all $t \in [0, T]$, and since $v(0) < 0$, hence $v(T) < 0$ which is at variance with $v(T) = 0$. Consequently conditions (23) are satisfied for all $t \in [0, t_2]$.

The function $F(\sigma)$ satisfies the equation $F(\sigma)\,F'(\sigma) + \varphi(\sigma) = 0$ and $\lambda = 0$. Hence from

(6) and (15) and the positive definiteness of the matrix L, we have for all $t > 0$

$$v^{\cdot} < -\theta\,[F(\sigma)F'(\sigma) + \varphi(\sigma)]\,r^*\,[Px + q\varphi(\sigma)] - \tau\,[r^*x - \varphi(\sigma)/\mu]\varphi(\sigma) \leqslant 0$$

Thus the inequality

$$v^{\cdot}(t) < 0, \quad \forall t \in [0, t_2), \ t_2 \leqslant +\infty \tag{24}$$

is satisfied for all $\lambda \leqslant 0$, and if $t_2 < +\infty$ then $x(t) \in \Omega_0$ for all $t \geqslant t_2$. It follows from (24) and $v(0) < 0$ that $v(t) < 0$ when $t \in [0, t_2)$. From this and (16) it follows that $\sigma \in (\sigma_1, \sigma_2)$ for all $t \geqslant 0$.

It is thus proved that when $x(0) \in \Omega$, i.e. $v(0) < 0$, then $x(t) \in \Omega$ for all $t \geqslant 0$. We shall now show that $x(t) \to 0$ when condition (11) is satisfied.

Assume that for all $t \geqslant 0$ $x(t) \in \Omega \setminus \Omega_0$. Since $x(t)$ is bounded, hence the function $v(t)$ is also bounded for all $t \geqslant 0$. This and (24) imply the existence of the limit

$$\lim_{t \to +\infty} V(x(t)) = V_0 \tag{25}$$

and the set of ω-limit points of the trajectory $x(t)$ is not empty /5/. By Theorem 2.2.5 of /5/ a trajectory $x_\omega(t)$ exists consisting entirely of ω-limit points of the trajectory $x(t)$. It follows from (25) that $V(x_\omega(t)) \equiv V_0$ for all $t \geqslant 0$, which contradicts (24).

Consequently, there is an instant of time t_2 such that $x(t_2) \in \Omega_0$, hence $x(t) \in \Omega_0$ for all $t \geqslant t_2$. By the conditions of Theorem 1 the characteristic equation of the sliding mode (9) has roots with negative real parts. Hence solutions of the linear system (8) approach the equilibrium state $x = 0$ as $t \to +\infty$. This shows that the solution $x(t)$ of system (1) approaches the point $x = 0$ as $t \to +\infty$. Theorem 1 is proved.

The statement of Lemma 1 follows from Lemma 1.2.1 and Theorem 1.2.6 of /5/, and from the non-degeneracy of the transfer function $\chi(p)$ of system 1.

The algorithm proposed in the proof of Theorem 1 is an extension of the algorithms /6–9, 12–22/ based on the construction of a Lyapunov function of the Lur'e type

$$V_1(x) = x^*Hx + \theta\int_0^\sigma \varphi(\sigma)\,d\sigma$$

where $H = H^* > 0$.

Indeed, suppose all the conditions of Theorem 1 are satisfied when $\lambda = 0$, i.e. the matrix P of the input system is a Hurwitz matrix. Integrating Eq.(4) for $\lambda = 0$ and $F(\sigma)F'(\sigma) + \varphi(\sigma) = 0$, we obtain

$$F^2(\sigma) = 2\int_0^\sigma \varphi(\sigma)\,d\sigma + F^2(0)$$

Thus the function $V(x)$ (14), when $\lambda = 0$, is identical, apart from a constant, with $V_1(x)$. Let σ_1, σ_2 be the values of σ for which condition (6) is satisfied. When $\sigma = \sigma_i$ $(i = 1, 2)$, we have

$$\frac{1}{2}F^2(-0) = \int_0^{\sigma_1} \varphi(\sigma)\,d\sigma + \frac{H_0}{\theta}, \quad \frac{1}{2}F^2(+0) = \int_0^{\sigma_2} \varphi(\sigma)\,d\sigma + \frac{H_0}{\theta}$$

$$\frac{1}{2}F^2(\sigma) = -\int_0^\sigma \varphi(\sigma)\,d\sigma + \min_{i=1,2}\int_0^{\sigma_i} \varphi(\sigma)\,d\sigma + \frac{H_0}{\theta}$$

The region Ω obtained by Thoerem 1 is defined for $\lambda = 0$ by the inequalities

$$x^*Hx + \theta\int_0^\sigma \varphi(\sigma)\,d\sigma < \theta\min_{i=1,2}\int_0^{\sigma_i} \varphi(\sigma)\,d\sigma + H_0, \quad \sigma_1 < \sigma < \sigma_2 \tag{26}$$

and coincides exactly with the region obtained in /6–9, 12–22/ using the function $V_1(x)$.

We select for $\lambda = 0$ the matrix $L > 0$ so as to have $x^*Lx < \delta|x|^2$ for any $\delta > 0$. The frequency condition (12) is then of the following form: $\operatorname{Re}[(\tau + i\omega\theta)\chi(i\omega)] + \tau\mu^{-1} > 0$ and is the same as Popov's frequency condition /23/ used in /6–9, 12–24/.

Thus the algorithms in /6–9, 12–22/, based on the construction of a Lyapunov function of the Lur'e type, are contained in the proof of Theorem 1 when $\lambda = 0$.

The sector (6) in which the non-linearity graph of $\varphi(\sigma)$ must lie when $\sigma \in [\sigma_1, \sigma_2]$ belongs to the sector

$$h_1\sigma^2 \leqslant \varphi(\sigma)\,\sigma \leqslant h_2\sigma^2, \ h_2 \leqslant +\infty \tag{27}$$

in which system (1) is linearized for $\varphi(\sigma) = h\sigma$, is stable for any $h \in (h_1, h_2)$. Regions obtained in /6–9, 12–24/ are bounded along the σ axis in the interval $[\sigma_{h1}, \sigma_{h2}]$ in which $\varphi(\sigma)$ satisfies (27). If Theorem 1 is applied for the inherently unstable system (1), then $\lambda < 0$ and condition (6) is identical with (2) and Theorem 1 does not contain requirements of the type of (27) (sector (6) in that case is wider than sector (27)). Region Ω is not bounded on the σ axis by the interval $[\sigma_{h1}, \sigma_{h2}]$ and for a number of systems it is outside the limits of the interval indicated. This is confirmed by the example given below.

Thoerem 1 enables thus to obtain in a number of cases for $\lambda < 0$ a region Ω more exactly approximating to the true attraction region of the stationary solution of a system of form (1) than the estimates in /6—9, 12—24/.

Consider now a system of the form

$$z^{\cdot} = Az + b\varphi(\sigma), \quad \sigma^{\cdot} = c^{*}z + \rho\varphi(\sigma) \tag{28}$$

where A is a constant $n \times n$-matrix of an arbitrary spectrum, $b, c - n$ are vectors, $\rho < 0$ is a number, and the function $\varphi(\sigma)$ satisfies condition (2). System (28) can be reduced to the form (1). We shall formulate for system (28) a result that can be considered as some "limit" case of Theorem (1). We introduce the notation $D(p) = c^{*}(A - pI)^{-1}b, \Gamma = -c^{*}b$, and for some matrix $L = L^{*} > 0$, $\Psi_1(\zeta) = \{z : z^{*}Lz \leqslant \zeta\}$, $\Pi_1 = \{z : -\rho\varphi(-0) \leqslant c^{*}z \leqslant -\rho\varphi(+0)\}$, $L_1 = \sup\{\zeta : \Psi_1(\zeta) \subset \Pi_1\}$. We assume that $\Gamma > 0$.

Theorem 2. We assume the existence of a matrix $L = L^{*} > 0$ and a number $\lambda < 0$, $\theta > 0$ that satisfy the conditions

1) $A + \lambda I$ is a Hurwitz matrix;

2) for all $\omega \in (-\infty, +\infty)$ the inequality

$$\theta \operatorname{Re} D(i\omega - \lambda) - b^{*}[A^{*} + (\lambda - i\omega)I]^{-1}L[A + (\lambda - i\omega)I]^{-1}b \geqslant 0 \tag{29}$$

is satisfied;

3) $\sqrt{L_1} \leqslant \sqrt{\rho\theta}\inf_{\sigma\neq 0}|\varphi(\sigma)|$;

4) $L(A - \rho^{-1}bc^{*}) + (A^{*} - \rho^{-1}cb^{*})L \leqslant 0$;

5) for the solution $F(\sigma)$ of Eq.(4) with $\alpha = -\lambda\Gamma^{-1/2}$, $\beta = -\rho\Gamma^{-1/2}$ and $s_0 = 0$ which has the properties (5) the relation $F(-0) = -F(+0) = \sqrt{L_1(\lambda\theta)^{-1}}$ holds.

The following statements hold:

a) all solutions $z(t)$, $\sigma(t)$ of system (28) with initial conditions

$$(z(0), \sigma(0)) \in \Omega = \{z, \sigma : z^{*}Hz < {}^1/_2\theta F^2(\sigma), \ \sigma_1 < \sigma < \sigma_2\} \cup \{z : z^{*}Lz \leqslant L_1\}$$

satisfy the inclusion $(z(t), \sigma(t)) \in \Omega$ for all $t \geqslant 0$;

b) if besides the satisfaction of condition 1)—5) $A - \rho^{-1}bc^{*}$ is a Hurwitz matrix, any solution of system (28) with initial condition $(z(0)$ and $\sigma(0)) \in \Omega$ approaches the point $z = 0, \sigma = 0$ as $t \to +\infty$.

The existence of the matrix H is guaranteed by the satisfaction of the frequency condition (29) of Lemma 1.2.6 in /5/.

Example. The application of Theorem 2 will be shown using the example of a system of second-order equations (Bulgakov's second problem) /25, 26/.

$$T^2\xi^{\cdot\cdot} + U\xi^{\cdot} - k\xi = -\varphi(\sigma), \quad \sigma^{\cdot} = G^2\xi^{\cdot\cdot} + E\xi^{\cdot} + a\xi - l^{-1}\varphi(\sigma), \quad \varphi(\sigma) = \begin{cases} 1, & \sigma > 0 \\ -1, & \sigma < 0 \end{cases} \tag{30}$$

where $\varphi(\sigma)$ is a function of the form (2), G^2, $a > 0$, $l > 0$, $E > 0$ are constants of the controller, and T, U are constants of the controlled system. We will consider system (30) for $U = 0$ and $G^2 = 0$.

System (30) can be reduced to a system of the form (28), where

$$A = \begin{Vmatrix} 0 & kT^{-2} \\ 1 & 0 \end{Vmatrix}, \quad b = \begin{Vmatrix} -T^2 \\ 0 \end{Vmatrix}, \quad c = \begin{Vmatrix} E \\ a \end{Vmatrix}, \quad \rho = -\frac{1}{l} \tag{31}$$
$$\Gamma = \frac{E}{T^2}, \quad z = \begin{Vmatrix} \xi^{\cdot} \\ \xi \end{Vmatrix}$$

The sliding system (30) is defined by the equation

$$\xi^{\cdot\cdot} = -\frac{lE}{T^2}\xi^{\cdot} - \frac{al - k}{T^2}\xi \tag{32}$$

The matrix L is seclected in the form

$$L = \begin{Vmatrix} \varepsilon & 0 \\ 0 & (al - k)T^{-2}\varepsilon \end{Vmatrix}$$

where $\varepsilon > 0$ is some number. The matrix L is positive definite when

$$al - k > 0. \tag{33}$$

Condition 4) of Theorem 2 is then satisfied.

We shall determine the number L_1 so that the system of equations $\varepsilon[z_1^2 + (al-k)T^{-2}z_2] = L_1$, $Ez_1 + az_2 = l^{-1}$ $(z_1 = \xi^{\cdot}, z_2 = \xi)$ will have a unique solution (i.e. the set $z^*Lz = L_1$ must touch the boundary of the band $|c^*z| = -\rho$). We obtain

$$L_1 = \varepsilon l^{-2}\Lambda^{-1}, \ \Lambda = E^2 + a^2T^2(al-k)^{-1} \tag{34}$$

Condition 1) of Therem 2 is satisfied if

$$-\lambda > \sqrt{k}/T \tag{35}$$

and condition 2)

$$\operatorname{Re}[(Ep+a)(T^2p^2-k)^{-1}] - \varepsilon\left(|p|^2 + \frac{al-k}{T^2}\right)|T^2p^2-k|^{-2} \geqslant 0$$

holds when

$$\varepsilon = -T^2(E\lambda + a) > 0$$
$$2T^2a\lambda^2 + Ek\lambda - ak + (E\lambda + a)(al-k) > 0 \tag{36}$$

Selecting $\lambda = -aE^{-1} - \delta_\lambda$ ($\delta_\lambda > 0$ is some number), and when the inequalties

$$E\sqrt{k} < aT, \quad E^2l \leqslant 4aT^2 \tag{37}$$

are satisfied condition (29) is satisfied. Taking into acccount (34) we obtain

$$F(-0) = \frac{T\sqrt{E\delta_\lambda}}{l\sqrt{-\lambda}\sqrt{\Lambda}}$$

Condition 3) of Theorem 2 has the form

$$\delta_\lambda \leqslant l\Lambda/(ET^2) \tag{38}$$

Hence, when relations (33), (37), and (38) are satisfied all conditions of Thoerem 2 are satisfied.

The region Ω is bounded along the σ axis by the segment (σ_1, σ_2), where $\sigma_1 = -\sigma_2$ are zeros of the function $F(\sigma)$. In system (30) the function $\varphi(\sigma)$ is a relay function. System (3) is then integrable. Calculating $\sigma(t) = \sigma_1$ at the instant t_1 at which $\eta(t_1) = 0$ $(\sigma(0) = 0, \ \eta(0) = F(-0))$, we obtain

$$\sigma_1 = \frac{1-\alpha\beta}{\alpha^2}\ln(1+\alpha\eta_0) - \frac{\eta_0}{\alpha} \tag{39}$$

$$\alpha = (a + E\delta_\lambda)E^{-1}/T, \quad \beta = \frac{T}{l\sqrt{E}}, \quad \eta_0 = F(-0) \tag{40}$$

Let us now compare the interval $(\sigma_1, -\sigma_1)$ obtained with the interval $(-\sigma_h, \sigma_h)$ that is cut out on the σ axis by sector (27) from system (30). To do this we consider the linear system obtained from (30) for $\varphi(\sigma) = \varkappa\sigma$. That system is stable when $\varkappa > al/E$, i.e. $h_1 = al/E$, $h_2 = +\infty$. The non-linearity of $\varphi(\sigma)$ belongs to sector (27) when $\sigma \in (-E/(al), E/(al))$.

Substituting into (40) the numerical values of the parameters $E = 4$, $a = 3$, $l = 1$, $T = 15$, $k = 1$, $\delta_\lambda = 0.4$, we obtain $\sigma_1 = -1.563$. The value of σ_h is then equal to 1.333.

The region (26) calculated in /6–9, 12–24/ is bounded on the σ axis by the interval $(-1.333; 1.333)$. since the region Ω cuts out on the σ axis an interval $(-1.563; 1.563)$. Consequently, the estimate of Ω obtained using Theorem 2 of the true attraction region is more accurate for system (30) than that constructed using the algorithms of the Lyapunov function of the Lur'e type.

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Translated by J.J.D.

PMM U.S.S.R.,Vol.49,No.6,pp.694-699,1985
Printed in Great Britain

0021-8928/85 $10.00+0.00
Pergamon Journals Ltd.

ON APPROXIMATE METHODS OF ANALYSING CERTAIN SINGULARLY-PERTURBED SYSTEMS*

L.K. KUZ'MINA

A certain class of singularly-perturbed systems which have a variety of m-dimensional stationary positions is considered. When a small parameter disappears, the system also has an m-dimensional manifold of stationary positions and, therefore, the corresponding characteristic equation has m zero roots. The conditions under which the solution of a stability problem reduces to the same problem for a degenerate system are defined. As an application in practice gyroscopic stabilizing systems (the critical case corresponds to such systems) with elastic elements of high stiffness are discussed. The conditions under which the solution of the problem of the stability of steady motion follows from the solution of this problem for an ideal system (with absolutely rigid elements) are obtained. The problem of the closeness of the corresponding solutions of the complete and a simplified system of differential equations over an infinite time interval is discussed.

1. Suppose the perturbed motion of a system is described by a differential equation of the form

**Prikl.Matem.Mekhan.*,49,6,909-915,1985